From global scaling, *à la* Kolmogorov, to local multifractal scaling in fully developed turbulence

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In the first part of the paper a modern presentation of scaling ideas is made. It includes a reformulation of Kolmogorov's 1941 theory bypassing the universality problem pointed out by Landau and a presentation of the multifractal theory with emphasis on scaling rather than on cascades. In the second part, various historical aspects are discussed. The importance of Kolmogorov's rigorous derivation of the $-\frac{4}{5}\epsilon l$ law for the third order structure function in his last 1941 turbulence paper is stressed; this paper also contains evidence that he was aware of universality not being essential to the 1941 theory.

An inequality is established relating the exponents ζ_{2p} of the structure functions of order $2p$ and the maximum velocity excursion. It follows that models (such as the Obukhov–Kolmogorov 1962 log-normal model), in which ζ_{2p} does not increase monotonically, are inconsistent with the basic physics of incompressible flow. This result is independent of Novikov's 1971 inequality; in particular, the proof presented here does not rely on the (questionable) relation, proposed by Obukhov and Kolmogorov, between instantaneous velocity increments and local averages of the dissipation.

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1. Introduction

The statement that turbulence remains an unsolved problem can hardly be debated. Yet, there is no consensus on how the problem of turbulence should be formulated. Half a century after Kolmogorov's work on the statistical theory of fully developed turbulence, we still wonder how his work can be reconciled with Leonardo's half a millennium old drawings of eddy motion in the study for the elimination of rapids in the river Arno. Here, I shall not even attempt to face this challenge. My intention is to concentrate on one aspect of Kolmogorov's work, namely scaling in fully developed turbulence. In the first part of this paper (§§1–4), I shall present a 'modern' viewpoint of turbulence. Historical aspects and proper crediting will be postponed to the second part (§§5 and 6). I hope to make it clear that the 'modern' viewpoint is, to a large extent, contained in Kolmogorov's manifold work of 1941. Yet my vision of the essence of that work is somewhat non-traditional. My frequent use of the first person in this essay is only meant to stress that I take full responsibility for my 'revisionist' perception of Andrei Nikolaevich. Through Batchelor's (1990) contribution to his obituary and through the contributions of my colleagues in this volume, the reader will be able to put together a more balanced view.

Section 2 is about global scaling and lack of universality. Section 3 is about local, multifractal scaling and its implications. Section 4 contains a new exact result

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about the exponents of structure functions. Section 5 is about Kolmogorov's (1941; hereafter referred to as K41) work and Landau. Section 6 is about intermittency and Kolmogorov. No conclusions will be presented.

2. Global scaling, *à la* Kolmogorov, and lack of universality

The Navier–Stokes equations for incompressible fluid flow possess a number of symmetries (invariance groups). When boundaries are ignored, the symmetries include: space and time translations, rotations, parity (space and velocity reversal) and galilean transformations. If the viscosity $\nu = 0$, an infinite class of additional symmetries appears, the scaling transformations:

$$\boldsymbol{r} \rightarrow \lambda \boldsymbol{r}, \quad \boldsymbol{v} \rightarrow \lambda^{h} \boldsymbol{v}, \quad t \rightarrow \lambda^{1-h} t, \quad \lambda \in \boldsymbol{R}_{+}. \tag{1}$$

Here, t, $\boldsymbol{r}$ and $\boldsymbol{v}$ are, respectively, the time, position and velocity variables. It is assumed that pressure has been eliminated from the Navier–Stokes equation through use of the incompressibility constraint. The different scaling groups are labelled by the scaling exponent $h \in \boldsymbol{R}$.

When a turbulent flow is set up in the laboratory, the production invariably involves mechanisms not consistent with some or all of the above symmetries. For example the presence of a rigid wall completely breaks the Galilean invariance and partially the translational invariance (perpendicular to the wall). Of course, the production mechanisms may be consistent with some symmetries. Non-moving boundaries are consistent with time translations; uniform flow incident upon a sphere is consistent with rotations around the diameter parallel to the flow, etc. An experimentally well-established fact is that when the control parameter (say, the Reynolds number) is increased, bifurcations occur which spontaneously break any such surviving symmetries. For example, a Hopf bifurcation may occur which turns the continuous time-translation symmetry into a discrete one. Eventually, after a suitable number of bifurcations, well-controlled flows are found to become chaotic. The flow then possesses a strange attractor. The continuous time-translation symmetry has thus been restored in a statistical sense, i.e. there is a measure in the phase space of the flow which is invariant under arbitrary time-translations. Our reformulation of K41 assumes that this statistical restoration of symmetries is not limited to time-translations, provided the infinite Reynolds number limit is taken, so as to allow the presence of motion on arbitrary small scales. Following Kolmogorov, I shall present the reformulation in the form of numbered hypotheses.

H1. *In the limit of infinite Reynolds numbers, all the possible symmetries of the Navier–Stokes equation, usually broken by the mechanisms producing the turbulent flow, are restored in a statistical sense at small scales and away from boundaries.*

The words 'small scales' can be technically defined by considering velocity increments over a distance l small compared to the integral scale l_0:

$$\delta \boldsymbol{v}(\boldsymbol{r}, \boldsymbol{l}) = \boldsymbol{v}(\boldsymbol{r}+\boldsymbol{l}) - \boldsymbol{v}(\boldsymbol{r}). \tag{2}$$

We may then define, for example, statistical invariance under space-translations (homogeneity) by:

$$\delta \boldsymbol{v}(\boldsymbol{r}+\boldsymbol{q}, \boldsymbol{l}) \stackrel{\mathrm{L}}{=} \delta \boldsymbol{v}(\boldsymbol{r}, \boldsymbol{l}), \quad q \ll l_0, \tag{3}$$

where $\stackrel{\mathrm{L}}{=}$ means 'equality in law' (identical statistical properties).

Since there is an infinity of different possible scaling exponents h, additional assumptions are needed.

H2. *Under the same assumptions as in H1, the turbulent flow is assumed to be self-similar at small scales, i.e. to possess a single scaling exponent h.*

The value of h is obtained from

H3. *Under the same assumptions as in H1, the turbulent flow is assumed to have a finite non-vanishing mean rate of dissipation ϵ per unit mass.*†

From H2 and H3, the value of the scaling exponent can be readily obtained. Indeed, Kolmogorov (1941*c*) has derived the following relation from the Navier–Stokes equation, under the sole assumptions of homogeneity, isotropy and finite mean energy dissipation:

$$S_3(l) \equiv \langle (\delta v_{\parallel}(\boldsymbol{r}, l))^3 \rangle = -\tfrac{4}{5}\epsilon l. \tag{4}$$

Here, $\delta v_{\parallel}$ denotes the component of the velocity increment parallel to the displacement vector l. The function S_3 is called the third order (longitudinal) structure function. The increment l is assumed by Kolmogorov to be small compared to the integral scale l_0. With the assumption H2, under rescaling of the increment l by a factor λ, the left-hand side of (4) changes by a factor λ^{3h} while the right-hand side changes by a factor λ. Hence,

$$h = \tfrac{1}{3}. \tag{5}$$

Under the assumption that moments of arbitrary integer order p of the velocity increment exist (there is considerable experimental evidence for this assumption), the self-similarity hypothesis implies scaling laws for structure functions of arbitrary order:

$$S_p(l) \equiv \langle (\delta v_{\parallel}(\boldsymbol{r}, l))^p \rangle = C_p \epsilon^{\frac{1}{3}p} l^{\frac{1}{3}p}. \tag{6}$$

The presence of the factors $\epsilon^{\frac{1}{3}p}$ in the right-hand side ensures that the C_ps are dimensionless. The C_ps cannot depend on the Reynolds number, since the limit of infinite Reynolds number is assumed. For $p = 3$, it follows from (4) that $C_3 = -\frac{4}{5}$, which is clearly universal. All the C_ps, except for $p = 3$, must, however, depend on the detailed geometry of the production of turbulence. In other words, they cannot be universal.

The non-universality of the C_ps is a central question in the reappraisal of K41 and will thus be discussed in some detail. The ϵ appearing in (6) is a mean dissipation rate, the mean being taken over the attractor of the flow, that is the mean is a time-average. Let us now construct a superensemble, made of $N > 1$ experiments possessing different values of the mean dissipation rate, denoted ϵ_i ($i = 1, \dots, N$). The differences could be caused, for example, by the flows having different integral scales. Let us tentatively assume that the C_ps are universal. We denote by $S_p^i(l)$ the structure function for the ith flow. We have, by (6),

$$S_p^i(l) = C_p (\epsilon_i)^{\frac{1}{3}p} l^{\frac{1}{3}p}. \tag{7}$$

† H3 is made plausible by the observation that the drag coefficient for flow past a body, which is related to the energy dissipation, is approximately independent of the Reynolds number over a wide range of values (Landau & Lifshitz 1987, §45). H3 is also compatible with results of numerical simulations at Reynolds numbers up to a few thousands. It is incorrect to infer H3 from the observation that in a statistical steady state, the mean energy dissipation equals the mean energy injection. Indeed, the latter is not controlled externally, except in the unrealistic case where injection is through a random force with white-noise time dependence. It is my feeling that H3 leaves considerable room for questioning.

Now, let us assume that it is legitimate to apply (6) to the superensemble (we shall come back to this). We define

$$S_p^{\text{super}}(l) = \frac{1}{N}\sum_i S_p^i(l) \quad \text{and} \quad \epsilon^{\text{super}} = \frac{1}{N}\sum_i \epsilon_i, \tag{8}$$

the superaveraged structure functions and dissipation rate respectively. From (6) and (7), we obtain:

$$\frac{1}{N}\sum_i (\epsilon_i)^{\frac{1}{3}p} = \frac{1}{N}\sum_i (\epsilon_i)^{\frac{1}{3}p}. \tag{9}$$

This relation is contradictory, except for $p = 3$.

The preceding argument depends crucially on the ability to consider the different flows as being part of a single superflow. This can be justified by considering a single flow in which the characteristic parameters change slowly in space on a scale much larger than the integral scale. Let us for example consider a wind tunnel in which a uniform flow of speed U is incident on a grid made of parallel rods with a uniform mesh m. There are two types of rods; type A has diameter d_1 and type B has diameter $d_2 > d_1$. In assembling the grid, type A and type B are selected at random in such a way that the type is changed on average every M rods, where M is a large number (say, 1000). The turbulence downstream (say, 100 meshes) behind type B rods has a larger integral scale than that behind type A rods. Hence, the dissipation rate per unit mass ϵ_B behind type B rods is smaller than the dissipation rate ϵ_A behind type A rods. (For dimensional reasons, ϵ scales as U^3/d.) In this example it is clear that the properties of the turbulent eddies at a given location can be significantly affected only by those rods behind which they are produced. Still, all the parts of the flow are coupled (for example by pressure effects), so that it is legitimate to treat the superensemble as a single flow.

3. Local, multifractal scaling

One possible weakness of the global scaling theory of §2 is that it ignores all the scaling symmetries not having $h = \frac{1}{3}$. An alternative is provided by multifractal scaling, in which, instead of H2, one uses

H_{mf}. *Under the same assumptions as in H1 (of §2), the turbulent flow is assumed to possess a range of scaling exponents* $I = (h_{\min}, h_{\max})$. *For each h in this range, there is a set* $\mathscr{S}_h \in \boldsymbol{R}^3$ *of Hausdorff dimension* $D(h)$, *such that, as* $l \to 0$

$$\delta v(\boldsymbol{r}, l) \propto l^h, \quad \boldsymbol{r} \in \mathscr{S}_h. \tag{10}$$

Assumptions H1 and H3 are kept unchanged.

The statement about the set $\mathscr{S}_h$ and its dimension $D(h)$ should be understood in a probabilistic sense: the probability of finding the scaling exponent h when varying the scale l at which the flow is observed is proportional to $l^{3-D(h)}$. (It takes $O(l^{-D})$ balls of radius l to cover a set of dimension D. Together, they cover a volume $O(l^{3-D})$.)

Expressions for structure functions of arbitrary order p are now easily derived. It is convenient to non-dimensionalize spatial increments l and velocity increments δv by the integral scale l_0 and the root mean square (r.m.s.) velocity fluctuation v_0

respectively. The symbol $\sim$ is then used to denote order one constants (mostly not universal). From H_{mf}, we obtain;

$$S_p(l)/v_0^p \equiv \langle(\delta v_{\parallel}(l))^p\rangle/v_0^p \sim \int_I \mathrm{d}\mu(h)\,(l/l_0)^{ph+3-D(h)}. \tag{11}$$

The exponent ph comes from the contribution of scaling with exponent h; the additional exponent $3-D(h)$ comes from the averaging process. The argument $\boldsymbol{r}$ has been omitted in δv, because of homogeneity. The measure $\mathrm{d}\mu(h)$ corresponds to the weight of the different scalings. The weight is not known but this does not affect the scaling properties of structure functions. Indeed, as $l \rightarrow 0$, a steepest descent argument indicates that, of all the exponents $ph+3-D(h)$, the smallest one dominates. Hence

$$S_p(l)/v_o^p \sim (l/l_0)^{\zeta_p}, \quad \zeta_p = \min_h(ph+3-D(h)). \tag{12}$$

Thus, the scaling exponents ζ_p of the structure function of order p is the Legendre transform of the dimension function $D(h)$. Since the inverse of a Legendre transform is also a Legendre transform, one obtains easily:

$$3-D(h) = \max_p(\zeta_p - ph). \tag{13}$$

Kolmogorov's relation (4) for the structure function of third order becomes simply:

$$\zeta_3 = \min_h(3h+3-D(h)) = 1. \tag{14}$$

4. An inequality for the exponents of structure functions

We shall show here that consistency with the basic physics of incompressible flow requires that the exponent ζ_{2p} of the structure function of order $2p$ should not decrease with p. The only assumptions made for the proof are the following. (i) In the limit $R = l_0 v_0/\nu \rightarrow \infty$, the structure functions of even order $2p$ possess the scaling exponents ζ_{2p}, that is, for $l \rightarrow 0$, one has to leading order:

$$\langle(\delta v_{\parallel}(l))^{2p}\rangle/v_0^{2p} = A_{2p}(l/l_0)^{\zeta_{2p}}, \tag{15}$$

where A_{2p} is a positive numerical constant (not necessarily universal). (ii) For large finite R, the scaling (15) still holds, as intermediate asymptotics, over a range of scales (inertial range) increasing with R at least as a power law:

$$1 \gg l/l_0 \gg R^{-\alpha}, \quad \alpha > 0. \tag{16}$$

We now establish two propositions.

Proposition 1. *Under the assumptions* (i), *if there exists two consecutive even numbers* $2p$ *and* $2p+2$ *such that*

$$\zeta_{2p} > \zeta_{2p+2}, \tag{17}$$

then the velocity of the flow (*measured in the reference frame of the mean flow*) *cannot be bounded.*

Proposition 2. *Under the assumptions* (ii) *and those of Proposition* 1, *if the Mach number based on* v_0 *is held fixed, and the Reynolds number is increased indefinitely*†, *then the maximum Mach number of the flow also increases indefinitely.*

† For example, by considering a sequence of grid-generated turbulent flows with ever-increasing mesh, all using the same fluid and the same flow velocity.

Proof. Let us denote by $U_{\max}$ the maximum velocity, taken over space and time. We have, at any instant of time

$$|\delta v_{\|}(\boldsymbol{r}, l)| \leqslant 2U_{\max}, \quad \forall \boldsymbol{r}, l. \tag{18}$$

The average being over time, it follows from (18) that

$$\langle(\delta v_{\|}(l))^{2p+2}\rangle \leqslant 4U_{\max}^2 \langle(\delta v_{\|}(l))^{2p}\rangle. \tag{19}$$

Assuming $l \ll l_0$ and using (15), we obtain

$$U_{\max}^2/v_0^2 \geqslant \tfrac{1}{4}(A_{2p+2}/A_{2p})\,(l/l_0)^{-(\zeta_{2p}-\zeta_{2p+2})}. \tag{20}$$

Using (17) and letting $l \to 0$, we find that $U_{\max} = \infty$. This establishes Proposition 1. We now define

$$M_0 = v_0/c_s, \quad M_{\max} = U_{\max}/c_s, \tag{21}$$

which are respectively the Mach number based on the r.m.s. velocity and on the maximum velocity (in the frame of the mean flow). We select a scale l by

$$l/l_0 = R^{-\frac{1}{2}\alpha}, \tag{22}$$

which by (16) is in the inertial range. Substituting (22) into (20) and using (21), we obtain:

$$M_{\max}^2 \geqslant \tfrac{1}{4}(A_{2p+2}/A_{2p})M_0^2 R^{\frac{1}{2}(\zeta_{2p}-\zeta_{2p+2})\alpha}. \tag{23}$$

Proposition 2 follows readily. *QED*.

A Mach number, measured in the reference frame of the mean flow, which becomes arbitrarily large violates a basic assumption needed in obtaining the incompressible Navier–Stokes equation.

In deriving the above propositions, we have assumed scaling for structure functions ((i) and (ii)), but not any Kolmogorov-type assumption such as H1, H2 and H3, or multifractal assumption such as H_{mf}. Still, let us observe that if we accept the multifractal formalism, then (17) implies (by (13)) the presence of negative scaling exponents h and thus again unbounded velocities at small scales.

5. Kolmogorov and Landau

Kolmogorov wrote three papers on turbulence in 1941 (Kolmogorov 1941*a–c*). In the first paper, the derivation of the $\frac{2}{3}$ law for the second order structure function is done via his first and second hypotheses of similarity. The first hypothesis states that inertial range and dissipation range statistical properties are uniquely and universally determined by ν and ϵ. The second hypothesis states that the inertial range statistical properties are uniquely and universally determined by ϵ. Kolmogorov does not explicitly derive expressions for structure functions of order p higher than two. It is however a straightforward consequence of his hypotheses that their inertial range behaviour is given by (6) with universal constants C_p.

In 1942 Kolmogorov presented his work at a seminar in the city of Kazan (on the Volga). Lev Landau was present and made a remark. What exactly he told Kolmogorov we can only try to reconstruct from the footnote which was inserted in the first (Russian) edition of the book on fluid mechanics Landau was writing with Evgeni Lifshitz, which appeared in 1944. In later editions this footnote found its way to the main text. It is worth quoting the full text of the remark. I am taking the

English from the most recent version of the book (Landau & Lifshitz 1987) and substituting my own notation for velocity increments, structure functions and integral scale.

> It might be thought that the possibility exists in principle of obtaining a universal formula, applicable to any turbulent flow, which should give $S_2(l)$ for all distances l that are small compared with l_0. In fact, however, there can be no such formula, as we see from the following argument. The instantaneous value of $(\delta v_{\parallel}(l))^2$ might in principle be expressed as a universal function of the dissipation ϵ at the instant considered. When we average these expressions, however, an important part will be played by the manner of variation of ϵ over times of the order of the periods of the large eddies (with size $\sim l_0$), and this variation is different for different flows. The result of the averaging therefore cannot be universal.

Landau's remark has been interpreted in many different ways. Often it is taken as an argument in favour of intermittency at scales small compared with the integral scale. I shall come back to this aspect in §6. My viewpoint is that Landau was concerned only with large scales. The essence of Landau's argument, as explained by Kraichnan (1974), is that 'the constant C_2 is not invariant to the composition of sub-ensembles...'. Landau formulated his argument in the temporal domain, but it can equally be recast in the spatial domain, as I have done in §2.

I stress that Landau's argument in no way rules out the $\frac{2}{3}$ law, but just the universality of the constant in front of $(\epsilon l)^{\frac{2}{3}}$. (A Landau-type argument has also been used to show the existence of intermittency in the far dissipation range (Kraichnan 1967) and its non-universality (Frisch & Morf 1981).) The presentation I used in §2 bypasses Landau's objection, because it postulates scale-invariance rather than universality. The hypothesis H1 about basic symmetries of the Navier–Stokes equation being recovered at small scales, may be found in the 1981 Les Houches lectures (Frisch 1983). Earlier, Orszag (1966) observed that the hierarchy of cumulant equations possesses scale-invariant solutions with (what amounts to) scaling exponent $\frac{1}{3}$.

Actually, Kolmogorov himself was probably aware of the crucial role of scale-invariance. To support this statement, let us consider the third 1941 paper (Kolmogorov 1941*c*). In this paper he begins by deriving (4) from the Kármán–Howarth equation. He makes the assumptions of homogeneity, isotropy and of finite non-vanishing energy dissipation (H3 of §2). Scale-invariance is not used. A full-derivation may be found in §34 of Landau & Lifshitz (1987). I consider this 'four fifth' law as perhaps the most important rigorous result in fully developed turbulence. After this derivation Kolmogorov makes the following statement (translated to my notation).

> It is natural to assume that for large l the ratio $S_3(l)/(S_2(l))^{\frac{3}{2}}$, i.e. the skewness of the distribution of probabilities for the difference $\delta v_{\parallel}(l)$ remains constant.

(In the context of the paper 'large' means at inertial range scales.) In other words, Kolmogorov *postulates a particular form of scale-invariance*. Also, notice that he assumes that the skewness is 'constant' (independent of scale) rather than 'universal' (independent of the flow). From this assumption and (4) he then recovers the $\frac{2}{3}$ law for $S_2(l)$ (his relation 9) and observes that 'in Kolmogorov (1941*a*) the relation (9) was deduced from somewhat different considerations'. This is why I believe that

Kolmogorov was aware of the existence of an alternative formulation to his early 1941 theory, not requiring universality assumptions. It is therefore legitimate to refer to the scale-invariant theory as 'K41'. Nevertheless, after Landau's remark, Kolmogorov did not try to salvage his 1941 theory and actually seems to have used the remark to develop (together with Obukhov) a new theory which takes into account intermittency effects.

6. Kolmogorov and intermittency

In 1961, at the Colloque International de Mécanique de la Turbulence in Marseilles Kolmogorov (1962) presented the so-called *log-normal* theory of intermittency, an outgrowth of previous work by Obukhov (1962). Landau is given considerable credit:

> But quite soon after they (the K41 hypotheses) originated, Landau noticed that they did not take into account a circumstance which arises directly from the assumption of essentially accidental and random character of the mechanism of transfer of energy from the coarser vortices to the finer: with increase of the ratio l_0/l, the variation of the dissipation of energy
>
> $$\epsilon = \frac{\nu}{2}\sum_{\alpha}\sum_{\beta}\left(\frac{\partial u_\alpha}{\partial x_\beta}+\frac{\partial u_\beta}{\partial x_\alpha}\right)^2 \tag{24}$$
>
> should increase without limit.

Curiously, in Landau's remark (as quoted in the previous section), I find no reference to 'fine' scales. Still, it has become a tradition to accept Kolmogorov's view crediting Landau (see for example the discussion in §25.1 of Monin & Yaglom (1975)). As I have shown, Landau's remark in no way implies that the K41 theory (in its scale-invariant version) is inconsistent. It may be that Landau communicated to Kolmogorov more than he put in the above quoted footnote. (I would be grateful to obtain any information on this matter.) My impression is rather that Landau's original remark triggered some independent thinking of Kolmogorov. Actually, Kolmogorov had been interested in the log-normal law already in 1941 when he proposed an interpretation for the approximate lognormality of the distribution of sizes in the process of pulverisation of mineral ore (Kolmogorov 1941*d*). He described this process as a cascade, the similarity of which to the Richardson cascade must have been obvious to him or become so at some point. (No reference to Richardson is made in the 1941 turbulence papers, but in the 1962 paper Kolmogorov writes that the K41 hypotheses 'were based physically on Richardson's idea of the existence in the turbulent flow of vortices of all possible scales...'.)

The Obukhov–Kolmogorov 1962 theory leads to serious difficulties some of which have not previously been reported. A central role is played by the spatial average of the energy dissipation over a ball of radius l centred at the point $\boldsymbol{r}$:

$$\epsilon_l(\boldsymbol{r}) \equiv \frac{1}{\frac{4}{3}\pi l^3}\int_{|\boldsymbol{r}'-\boldsymbol{r}|<l} \mathrm{d}^3 r' \,\frac{1}{2}\nu \sum_{ij} (\partial_j v_i(\boldsymbol{r}') + \partial_i v_j(\boldsymbol{r}'))^2. \tag{25}$$

Kolmogorov's key hypotheses, in slightly reformulated form, are as follows.

K62*a*. The logarithm of ϵ_l has a gaussian (normal) distribution with variance

$$\sigma_l^2 = A + \mu \ln(l_0/l), \tag{26}$$

where μ is a positive adjustable parameter.

K62*b*. The scaling properties of the fluctuating velocity increment over a distance l are related to the scaling properties of the fluctuating dissipation by

$$(\delta v_{\parallel}(\boldsymbol{r}, l))^3/l \stackrel{s}{=} \epsilon_l(\boldsymbol{r}), \tag{27}$$

where the symbol $\stackrel{s}{=}$ is used to denote that the left- and the right-hand sides have the same scaling properties, i.e. that moments of the same order have the same scaling exponents.

As is well-known, it follows from K62*a* and K62*b* that the scaling exponents for the structure function of order p are given by

$$\zeta_p = \tfrac{1}{3}p - \tfrac{1}{18}\mu p(p-3). \tag{28}$$

Novikov (1971) observed that for fixed $\boldsymbol{r}$, the quantity $l^3\epsilon_l(\boldsymbol{r})$ is a non-decreasing function of l and deduced from this that for large p, the correction to the K41 value of ζ_p cannot grow faster than linear, thereby contradicting the parabolic behaviour predicted by the log-normal model. Actually, I have shown in §4 without recourse to either K62*a* nor K62*b* that a function ζ_{2p} which decreases with p, as is the case at large ps in the log-normal model, violates the basic physics of incompressible flow.

The log-normal hypothesis K62*a* has been frequently questioned (see, for example, in addition to Novikov (1971), Mandelbrot (1968, 1974*b*, 1976), Kraichnan (1974)). The hypothesis K62*b* relating instantaneous velocity increments and local dissipation seems to be more widely accepted. It should not be. Kraichnan (1974) observes that the left-hand side of (27) is an inertial range quantity while the right-hand side is a mixed dissipation range and inertial range quantity (the largest contribution to the rate-of-strain comes from dissipative scales, while the integration in (25) is over inertial distances). I now observe that Kolmogorov's four-fifth law (4) implies the truth of the relation obtained by taking the moment of order $q = 1$, i.e. the average of (27). (Subject, of course, to the same rather weak assumptions made in deriving (4).) For moments of order $q \neq 1$ three causes of suspicion can be raised. The first is that $l^3\epsilon_l$, being a space-integral, is an additive quantity (if one-dimensional rather than three-dimensional space averages are used the additive quantity is $l\epsilon_l$); similarly, $\delta v_{\parallel}(\boldsymbol{r}, l)$ is an additive quantity (if A, B and C are three consecutive points on a line, the longitudinal velocity difference between points A and C is the sum of the difference between A and B and the difference between B and C; however, the cube of an additive quantity is not additive. The second cause of suspicion is that $\delta v_{\parallel}(\boldsymbol{r}, l)$ fluctuates around a zero mean value; thus negative moments of order $q \leqslant -1$ are generally infinite; in contrast ϵ_l is positive and may have negative moments if its probability distribution vanishes sufficiently fast near the value zero (Bacry *et al.* 1990). The third cause of suspicion is that (27) correctly predicts structure functions of order $p > 1$ for Burgers model (they are dominated by the contributions of shocks) but incorrectly predicts those of order $0 \leqslant p < 1$ which are dominated by the contributions of non-dissipative velocity-ramps (Aurell *et al.* 1991).

It is paradoxical that despite all the aforementioned difficulties with Kolmogorov's 1962 paper, it nevertheless led to many fruitful further developments. Theoretical developments were mostly concerned with intermittent cascade models. Other papers in this issue are dealing with this question. I shall here discuss only aspects in which I was directly involved. In the late sixties Mandelbrot observed that multiplicative cascade models, when continued indefinitely, lead to an energy dissipation generally concentrated on a set of non-integer Hausdorff dimension (Mandelbrot 1968, 1974*b*). I was fascinated by this idea of 'fractal' turbulence and later with my colleagues Sulem and Nelkin, we tried to reconcile this viewpoint and Kraichan's observation that in deriving inertial range scaling one should work exclusively with inertial range quantities (Frisch *et al.* 1978). The simple 'β-model' was thus constructed by a suitable reformulation of the Novikov & Stewart (1964) model, stressing its dynamical and fractal aspects. This model became perhaps excessively popular. Indeed, the β-model was intended to be a minimally complex toy model and not a predictive model.

A few years later, Anselmet *et al.* (1984) obtained experimental data on high-order structure functions of far better quality than previously feasible. Not surprisingly, the values of the scaling exponents ζ_p agreed neither with the β-model nor with the log-normal model. Their figure 14 indicates that ζ_p increases with p over the whole accessible range of exponents but has a curvature not compatible with the linear-plus-constant behaviour of the β-model. Such a curvature is consistent with a general class of multiplicative cascade models which Mandelbrot (1974*b*, 1976) called 'weighted curdling models' because they do not have the black and white character of the Novikov–Stewart model. But here again, it was desirable to reinterpret the models in terms of pure inertial range quantities. This was done by Parisi & Frisch (1985) and led to the multifractal model as formulated in §3, in which a central role is played by the Legendre transformation. Contrary to the formulation of the β-model, the formulation of the multifractal model did not make use of the concept of cascade. Of course, a bridge can be established between the multifractal model and Mandelbrot's cascade models. This is best done via a large deviation argument discussed, for example, in Oono (1989, Appendix C). (Mandelbrot (1974*a*) already used this argument in a paper in which the equivalent of the Parisi–Frisch function $D(h)$ is introduced without being interpreted as a dimension.) In my view, the multifractal model is much closer to Kolmogorov's 1941 ideas because it explicitly embodies the idea of scaling (albeit in local form).

Finally, I mention that it was found recently that the multifractal model implies in a rather obvious way a prediction for the shape of the energy spectrum in the dissipation range (Frisch & Vergassola 1991). Due to the fact that the different scaling exponents h have different viscous cutoffs a new form of universality is predicted: $\log E(k)/\log R$ should be a function of universal shape of $\log k/\log R$. Because of the divisions by $\log R$, this is not consistent with Kolmogorov's first hypothesis of similarity. Experimental data analysed by Gagne & Castaing (1991) give good support to this 'multifractal universality' which also appears in thermal convection (Wu *et al.* 1990).

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